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Running Coupling Expansion for the Renormalized ϕ_4^4-Trajectory from Renormalization Invariance

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We formulate a renormalized running coupling expansion for the β-function and the potential of the renormalized ϕ^4-trajectory on four-dimensional Euclidean space-time. Renormalization invariance is used as a first principle. No reference is made to bare quantities. The expansion is proved to be finite to all orders of perturbation theory. The proof includes a large-momentum bound on the connected free propagator amputated vertices.

KEY WORDS: Renormalized ϕ^4-trajectory; running coupling; renormalization invariance; renormalized perturbation theory.

1. INTRODUCTION

We study the renormalization of a massless real scalar field ϕ on four dimensional space-time, perturbed by a ϕ^4-vertex. Its renormalization is done by means of a renormalization group transformation R_L, which scales by a factor $L > 1$. The renormalized theory comes as a pair, consisting of a β-function $\beta(g)$ together with a potential $V(\phi, g)$, both functions of the ϕ^4-coupling g (but not of L), with the following properties:

(I) $V(\phi, g)$ is of the form

$$V(\phi, g) = g \int d^4x \left\{ \frac{\mu^{(1)}}{2} \phi(x)^2 + \frac{\zeta^{(1)}}{2} \phi(x)(-\Delta)\phi(x) + \frac{1}{4!} \phi(x)^4 \right\} + O(g^2) \quad (1)$$

(II) $V(\phi, g)$ is invariant in the sense that

$$(R_L V_{g_1})(\phi) = V_{g(L)}(\phi) \quad (2)$$

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where $g(L)$ is the solution to the flow equation

$$L \frac{d}{dL} g(L) = \beta(g(L)) \tag{3}$$

with the initial condition $g_1 = g(1)$.

We will show that there exists a β-function $\beta(g)$ and a potential $V(\phi, g)$, both unique to all orders of perturbation theory in g, enjoying these properties (and general qualities of a renormalized potential in ϕ^4-theory). The assignment $g \mapsto \{\beta(g), V(\phi, g)\}$ is called the renormalized ϕ^4-trajectory.

The idea of analyzing the vicinity of renormalization fixed points in normal coordinates can be viewed as an application of Ecalle's theory for general dynamical systems. See the article [EW84] by Eckmann and Wittwer and references therein. We will treat perturbation theory in the sense of a formal power series. The analytic properties of its Borel transform will not be discussed. Work in this direction has been done by Gawedzki, Kupiainen, and Tirozzi [GKT85]. We will consider small perturbations of a free massless field, in other words the vicinity of the trivial fixed point. The perturbation theory will be done inductively as in Polchinski's proof of perturbative renormalizability [P84]. Unlike Polchinski we will not start from a bare action, and we will not consider renormalization as a mixed boundary value problem.

This paper extends the perturbation theory in [RW96] for the hierarchical approximation to the full model with momentum space cutoff. We will restrict our attention to the four dimensional case where no logarithms of the running coupling appear. The normal form of the β-function is cubic in four dimensions. Renormalization invariance was used as a first principle in [Wi96], both for a discrete and a continuous renormalization group. The aim of the present paper is to give a short, in the sense of formal power series rigorous, construction of a renormalized perturbation expansion in the continuous renormalization group. The setup will be close to that in [Wi96], with the difference that we will do without normal ordering here, and with the difference that we will admit a more general β-function.

2. RENORMALIZATION GROUP

We begin on a formal level, which is strengthened as we move towards perturbation theory.

Let ϕ be a real scalar field on four dimensional Euclidean space-time. Consider the following renormalization group for potentials $V(\phi)$, derived from a momentum space decomposition of $(-\Delta)^{-1}$.

Definition. Let R_L be the renormalization group transformation

$$(R_L \mathrm{V})(\psi) = -\log \int \mathrm{d}\mu_{\Gamma_L}(\zeta) \exp\{-\mathrm{V}(S_L \psi + \zeta)\} \tag{4}$$

depending on a scale parameter $L > 1$, where $\mathrm{d}\mu_{\Gamma_L}(\zeta)$ denotes the Gaussian measure on field space with mean zero and covariance

$$\widetilde{\Gamma_L}(p) = \frac{\exp(-p^2) - \exp(-L^2 p^2)}{p^2} \tag{5}$$

and where S_L denotes the the dilatation operator

$$\widetilde{S_L \psi}\,(p) = L^3 \tilde{\psi}(Lp). \tag{6}$$

Field independent constants are understood to be properly removed. Notice that ψ is rescaled with its canonical scaling dimension. The renormalization group transformation (4) is a Gaussian convolution in resealed form.

Concerning the background on the renormalization group, we refer to Wilson and Kogut [WK74]. A pedagocical account of the perturbative momentum space renormalization group can be found in the lectures by Benfatto and Gallavotti [BG95]. It was applied to the perturbative renormalization of QED by Feldman, Hurd, Rosen, and Wright [FHRW88].

Proposition 2.1. R_L satisfies the semi-group property

$$R_L \circ R_{L'} = R_{L L'}, \qquad L, L' > 1, \qquad \lim_{L \downarrow 1} R_L = id \tag{7}$$

Consequently, the iteration of (4) with fixed L is interpolated by an increase of L in one transformation (4).

Proposition 2.2. The renormalization group flow $\mathrm{V}(\psi, L) = (R_L \mathrm{V})(\psi)$ satisfies the functional differential equation

$$\begin{aligned} &\left\{ L \frac{\partial}{\partial L} - \left(D\psi, \frac{\delta}{\delta\psi} \right) \right\} \mathrm{V}(\psi, L) \\ &\quad = \left(\frac{\delta}{\delta\psi}, C \frac{\delta}{\delta\psi} \right) \mathrm{V}(\psi, L) - \left(\frac{\delta}{\delta\psi} \mathrm{V}(\psi, L), C \frac{\delta}{\delta\psi} \mathrm{V}(\psi, L) \right) \end{aligned} \tag{8}$$

where

$$D\tilde{\psi}(p)=\left\{p\frac{\partial}{\partial p}+3\right\}\tilde{\psi}(p),\qquad \tilde{C}(p)=\exp(-p^2) \tag{9}$$

with the initial condition $\mathrm{V}(\psi,1)=\mathrm{V}(\psi)$.

The continuous renormalization group was invented by Wilson [WK74]. A review of its applications was given by Wegner [We76]. Its value in perturbative renormalization was discovered by Polchinski [P84]. Functional differential equations for interpolated Gaussian convolutions are also used in the cluster expansion of Glimm and Jaffe [GJ87].

An aim of renormalization theory is to construct renormalization group flows which remain finite as $L\uparrow\infty$. A way to proceed is to look for quantities which are independent of L.

Definition. A scaling pair is a β-function $\beta(g)$ together with a potential $V(\psi, g)$, both depending on a coupling g but not on L, such that

$$\mathrm{V}(\psi, L)=V(\psi, g(L)) \tag{10}$$

satisfies (8) for any solution $g(L)$ of the ordinary differential equation

$$L\frac{\mathrm{d}}{\mathrm{d}L}g(L)=\beta(g(L)) \tag{11}$$

A scaling potential is its own renormalization image in the sense that

$$(R_L V_{g_1})(\psi)=V_{g(L)}(\psi) \tag{12}$$

where $g(L)$ is the solution of the one dimensional flow equation (11) to the initial condition $g(1)=g_1$. In view thereof, $g(L)$ is called a running coupling.

Proposition 2.3. A β-function $\beta(g)$ together with a potential $V(\psi, g)$ is a scaling pair if both together satisfy the functional differential equation

$$\begin{aligned}&\left\{\beta(g)\frac{\partial}{\partial g}-\left(D\psi,\frac{\delta}{\delta\psi}\right)\right\}V(\psi,g)\\&\quad=\left(\frac{\delta}{\delta\psi},C\frac{\delta}{\delta\psi}\right)V(\psi,g)-\left(\frac{\delta}{\delta\psi}V(\psi,g),C\frac{\delta}{\delta\psi}V(\psi,g)\right)\end{aligned} \tag{13}$$

We will restrict our attention to Euclidean invariant even potentials. Let $V(\psi, g)$ be given by a power series

$$V(\psi, g) = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \int \frac{d^4 p_1}{(2\pi)^4} \cdots \int \frac{d^4 p_{2n}}{(2\pi)^4} (2\pi)^4 \delta\left(\sum_{i=1}^{2n} p_i\right)$$
$$\times \tilde{\psi}(-p_1) \cdots \tilde{\psi}(-p_{2n}) \tilde{\psi}\left(\sum_{i=1}^{2n-1} p_i\right) \tilde{V}_{2n}(p_1, ..., p_{2n}, g) \tag{14}$$

in ψ. The question of its convergence will be left aside. Let us instead identify $V(\psi, g)$ with its formal sequence of vertices $\tilde{V}_{2n}(p_1, ..., p_{2n}, g)$. Vertices will be restricted to the hyperplane of total zero momentum. They can then be represented as

$$\tilde{V}_{2n'}\left(p_1, ..., p_{2n-1}, -\sum_{i=1}^{2n-1} p_i, g\right) = \tilde{V}_{2n}(p_1, ..., p_{2n-1}, g) \tag{15}$$

Proposition 2.4. A β-function $\beta(g)$ together with a potential $V(\psi, g)$, viewed as a formal power series in ψ, is a scaling pair if both together satisfy the system of integro-differential equations

$$\left\{\beta(g) \frac{\partial}{\partial g} + \sum_{i=1}^{2n-1} p_i \frac{\partial}{\partial p_i} - 4 + 2n\right\} \tilde{V}_{2n}(p_1, ..., p_{2n-1}, g)$$
$$= \int \frac{d^4 q}{(2\pi)^4} \tilde{C}(q)\, \tilde{V}_{2(n+1)}(p_1, ..., p_{2n-1}, q, -q, g)$$
$$- \sum_{m=1}^{n} \binom{2n}{2m-1} \left[\tilde{C}\left(\sum_{i=1}^{2m-1} p_i\right) \tilde{V}_{2m}(p_1, ..., p_{2m-1}, g)\right.$$
$$\left.\times \tilde{V}_{2(n-m+1)}\left(p_{2m}, ..., p_{2n-1}, \sum_{i=1}^{2m-1} p_i, g\right)\right]_{\mathfrak{S}_{2n-1}} \tag{16}$$

where $[\,\cdot\,]_{\mathfrak{S}_{2n-1}}$ denotes the symmetrization in $p_1, ..., p_{2n-1}$.

The constant $4-2n$ is called the scaling dimension of a vertex. Furthermore, vertices are called relevant, marginal, or irrelevant when their scaling dimension is positive, zero, or negative.

3. Φ^4-THEORY

The non-irrelevant couplings of $V(\psi, g)$ play a special role and deserve their own names. Let $\mu(g)$, $\zeta(g)$, and $\lambda(g)$ be defined as

$$\mu(g) = \tilde{V}_2(0, g), \quad \zeta(g) = \frac{\partial}{\partial(p^2)} \tilde{V}_2(p, g)\bigg|_{p=0}, \quad \lambda(g) = \tilde{V}_4(0, 0, 0, g). \tag{17}$$

A canonical choice of g in ϕ^4-theory is the value of the quartic vertex at zero momentum. We prefer a slightly more general definition.

Definition. Let $\lambda(g)$ be a given formal power series

$$\lambda(g) = g + \sum_{r=2}^{\infty} \frac{g^r}{r!} \lambda^{(r)} \tag{18}$$

in $g + g^2\mathbb{R}[[g]]$.

The normalization $\lambda^{(1)} = 1$ can always be achieved by a rescaling of g. The choice $\lambda^{(r)} = 0, r > 1$, means selecting the ϕ^4-coupling as expansion parameter. Other choices serve to bring the β-function to a standard form. The cubic normal form will be discussed below. Any choice will do for the moment. $\lambda(g)$ will now be assumed to be fixed.

We then expand both the β-function and the vertices into power series in g,

$$\beta(g) = \sum_{r=1}^{\infty} \frac{g^r}{r!} \beta^{(r)} \tag{19}$$

$$\tilde{V}_{2n}(p_1,\dots, p_{2n-1}, g) = \sum_{r=1}^{\infty} \frac{g^r}{r!} \tilde{V}_{2n}^{(r)}(p_1,\dots, p_{2n-1}) \tag{20}$$

Power series expansions for the couplings (17) are included. All of them will be treated as formal power series in g.

Proposition 3.1. A β-function $\beta(g)$ together with a potential $V(\psi, g)$, viewed as formal power series in g, is a scaling pair if both together satisfy the system of integro-differential equations

$$\begin{aligned}
&\left\{ \sum_{i=1}^{2n-1} p_i \frac{\partial}{\partial p_i} - 4 + 2n + r\beta^{(1)} \right\} \tilde{V}_{2n}^{(r)}(p_1,\dots, p_{2n-1}) \\
&\quad = - \sum_{s=2}^{r} \binom{r}{s} \beta^{(s)} \tilde{V}_{2n}^{(r-s+1)}(p_1,\dots, p_{2n-1}) \\
&\quad\quad + \int \frac{d^4q}{(2\pi)^4} \tilde{C}(q) \tilde{V}_{2(n+1)}^{(r)}(p_1,\dots, p_{2n-1}, q, -q) \\
&\quad\quad - \sum_{s=1}^{r-1} \binom{r}{s} \sum_{m=1}^{n} \binom{2n}{2m-1} \left[\tilde{C}\left(\sum_{i=1}^{2m-1} p_i \right) \tilde{V}_{2m}^{(s)}(p_1,\dots, p_{2m-1}) \right. \\
&\quad\quad \left. \times \tilde{V}_{2(n-m+1)}^{(r-s)} \left(p_{2m},\dots, p_{2n-1}, \sum_{i=1}^{2m-1} p_i \right) \right]_{\mathfrak{S}_{2n-1}}
\end{aligned} \tag{21}$$

To obtain a mathematically well defined problem, we should say what kind of solutions are looking for.

Definition. Let $\mathbb{V}$ be the space of vertices $\tilde{V}_{2n}^{(r)}(p_1,..., p_{2n-1})$ with the following properties:

(I) (Bose-symmetry)

$$\tilde{V}_{2n}^{(r)}(p_{\pi(1)},..., p_{\pi(2n)}) = \tilde{V}_{2n}^{(r)}(p_1,..., p_{2n}), \qquad \pi \in \mathfrak{S}_{2n} \tag{22}$$

(II) ($O(4)$-symmetry)

$$\tilde{V}_{2n}^{(r)}(Rp_1,..., Rp_{2n}) = \tilde{V}_{2n}^{(r)}(p_1,..., p_{2n-1}), \qquad R \in O(4) \tag{23}$$

(III) (Smoothness)

$$\tilde{V}_{2n}^{(r)} \in \mathscr{C}^{\infty}(\mathbb{R}^4 \times \cdots \times \mathbb{R}^4) \tag{24}$$

(IV) (Large momentum bound)

$$\begin{aligned} &\|\partial^{\alpha} \tilde{V}_{2n}^{(r)}\|_{\infty, \varepsilon} \\ &= \sup_{(p_1,..., p_{2n-1}) \in \mathbb{R}^4 \times \cdots \times \mathbb{R}^4} \left\{ |\partial^{\alpha} \tilde{V}_{2n}^{(r)}(p_1,..., p_{2n-1})| \exp\left(-\varepsilon \sum_{i=1}^{2n-1} p_i^2\right)\right\} < \infty, \\ &\qquad 0 < \varepsilon < \tfrac{1}{2}, \qquad \alpha \in \mathbb{N}^4 \times \cdots \times \mathbb{N}^4 \end{aligned} \tag{25}$$

(V) (Connectedness)

$$\tilde{V}_{2n}^{(r)}(p_1,..., p_{2n-1}) = 0, \qquad n > r+1 \tag{26}$$

(VI) (Coupling)

$$\tilde{V}_4^{(r)}(0, 0, 0) = \lambda^{(r)}, \qquad r > 1 \tag{27}$$

(VII) (Order one)

$$\tilde{V}_2^{(1)}(p) = \mu^{(1)} + \zeta^{(1)} p^2 \tag{28}$$

$$\tilde{V}_4^{(1)}(p_1, p_2, p_3) = 1 \tag{29}$$

$$\tilde{V}_{2n}^{(1)}(p_1,..., p_{2n-1}) = 0, \qquad n > 2 \tag{30}$$

The properties (I), (II), (III), and (IV) are appropriate for ϕ^{2N}-theory, with any $N > 1$, in this setup. The properties (V), (VI), and (VII) distinguish ϕ^4-theory. See also Polchinski [P84] and Keller, Kopper, and Salmhofer [KKS91].[2]

Theorem 3.2. (A) There exists a unique scaling pair in $\mathbb{V}$, given by β-function $\beta(g)$ together with potential $V(\psi, g)$, both viewed as formal power series in g, whose vertices $\tilde{V}_{2n}^{(r)}(p_1,..., p_{2n-1})$ have the properties (I),..., (VII). It is called the renormalized ϕ^4-trajectory. (B) The β-function of the renormalized ϕ^4-trajectory is given by $\beta(g) = (-3/(4\pi)^2)\, g^2 + O(g^3)$. The running coupling $g(L)$ is therefore asymptotically free in the infrared direction.

Outline of the Proof. The proof is an induction on r. The induction step $r-1 \to r$ consists of a sub-induction $n+1 \to n$, which goes backwards in the number of legs. We compute $\tilde{V}_{2n}^{(r)}(p_1,..., p_{2n-1})$ in the order $r+1, r,..., 1$. When coming to the case $n=2$, we first compute $\beta^{(r)}$ and thereafter $\tilde{V}_4^{(r)}(p_1, p_2, p_3)$. In the case $n=1$, we first compute the mass $\mu^{(r)} = \tilde{V}_2^{(r)}(0)$, then the wave function $\zeta^{(r-1)} = (\partial/\partial(p^2))\, \tilde{V}_2^{(r-1)}(p)|_{p^2=0}$, and thereafter $\tilde{V}_2^{(r)}(p)$, except for $\zeta^{(r)}$. Each of these steps will be shown to be both well defined and to yield a unique solution.

4. PROOF OF THE THEOREM

To first order, (21) simplifies to

$$\left\{\sum_{i=1}^{2n-1} p_i \frac{\partial}{\partial p_i} - 4 + 2n + \beta^{(1)}\right\} \tilde{V}_{2n}^{(1)}(p_1,..., p_{2n-1})$$
$$= \int \frac{d^4q}{(2\pi)^4}\, \tilde{C}(q)\, \tilde{V}_{2(n+1)}^{(1)}(p_1,..., p_{2n-1}, q, -q) \tag{31}$$

Lemma 4.1. The first order vertices, given by (28), (29), and (30), satisfy (31) if and only if

$$\beta^{(1)} = 0, \qquad \mu^{(1)} = \frac{-1}{2(4\pi)^2} \tag{32}$$

The first order coupling $\zeta^{(1)}$ is a free parameter.

[2] The authors use a cutoff function with compact support. The large momentum bound is then unneccessary as all loop integrals extend over a finite domain.

Proof. For $n=2$, (29) and (30) satisfy (31) if $\beta^{(1)}=0$. For $n=1$, (28) and (29) satisfy (31) if

$$\mu^{(1)}=\frac{-1}{2}\int\frac{d^4q}{(2\pi)^4}\,\tilde{C}(q) \tag{33}$$

This integral is convergent and evaluated to (32). ∎

$\mu^{(1)}$ is a normal ordering constant for the first order quartic vertex. $\zeta^{(1)}$ is better thought of as a second order quantity. Its value will be computed from a second order equation.

Hypothesis. Suppose that we have determined all coefficients $\beta^{(s)}$ and all vertices $\tilde{V}_{2m}^{(s)}(p_1,..., p_{2m-1})$, for $1\leqslant s\leqslant r-1$ and $1\leqslant m\leqslant s+1$, except for the coupling $\zeta^{(r-1)}$. Suppose further that we have determined $\tilde{V}_{2m}^{(r)}(p_1,..., p_{2m-1})$, for $n+1\leqslant m\leqslant r+1$. Suppose that all vertices, determined so far, have the properties (I),..., (VII). We proceed with the computation of $\tilde{V}_{2n}^{(r)}(p_1,..., p_{2n-1})$ under these assumptions.

To save space we write

$$\begin{aligned}\tilde{K}_{2n}^{(r)}(p_1,..., p_{2n-1}) = &-\sum_{s=2}^{r}\binom{r}{s}\beta^{(s)}\,\tilde{V}_{2n}^{(r-s+1)}(p_1,..., p_{2n-1})\\ &+\int\frac{d^4q}{(2\pi)^4}\,\tilde{C}(q)\,\tilde{V}_{2(n+1)}^{(r)}(p_1,..., p_{2n-1}, q, -q)\\ &-\sum_{s=1}^{r-1}\binom{r}{s}\sum_{m=1}^{n}\binom{2n}{2m-1}\Bigg[\tilde{C}\left(\sum_{i=1}^{2m-1}p_i\right)\\ &\times\tilde{V}_{2m}^{(s)}(p_1,..., p_{2m-1})\,\tilde{V}_{2(n-m+1)}^{(r-s)}\\ &\times\left(p_{2m},..., p_{2n-1}, \sum_{i=1}^{2m-1}p_i\right)\Bigg]_{\mathfrak{S}_{2n-1}}\end{aligned} \tag{34}$$

for the right hand side of (21).

Lemma 4.2. The integral in (34) is convergent. The differential vertex given by (34), has the properties (22), (23), (24), (25), and (26).

Proof. We prove the large momentum bound (25) for the case of no momentum derivatives. Use part of the expontential decay of $\tilde{C}(q)$ for an

$L_{\infty,\varepsilon}$-bound on $\tilde{V}^{(r)}_{2(n+1)}(p_1,..., p_{2n-1}, q, -q)$. Put an L_1-bound on the remaining one loop integral. The result is an estimate

$$\|\tilde{K}^{(r)}_{2n}\|_{\infty,\varepsilon} \leqslant \sum_{s=2}^{r-1} \binom{r}{s} |\beta^{(s)}| \, \|\tilde{V}^{r-s+1}_{2n}\|_{\infty,\varepsilon} + \frac{C_0}{(1-2\varepsilon)^2} \|\tilde{V}^{(r)}_{2(n+1)}\|_{\infty,\varepsilon}$$
$$+ \sum_{s=1}^{r-1} \binom{r}{s} \sum_{m=1}^{n} \binom{2n}{2m-1} \|\tilde{V}^{(s)}_{2m}\|_{\infty,\varepsilon} \, \|\tilde{V}^{(r-s)}_{2(n-m+1)}\|_{\infty,\varepsilon} \qquad (35)$$

where C_0 is a constant, independent of r and n. Momentum derivatives are distributed on all factors, which are then estimated along the same lines. The other assertions are elementary. ∎

Therefore, we have a well defined first order partial differential equation

$$\left\{ \sum_{i=1}^{2n-1} p_i \frac{\partial}{\partial p_i} - 4 + 2n \right\} \tilde{V}^{(r)}_{2n}(p_1,..., p_{2n-1}) = \tilde{K}^{(r)}_{2n}(p_1,..., p_{2n-1}) \qquad (36)$$

for the vertex labelled by n and r. The perturbative scaling dimension is $4-2n$, independent of r, whence $\beta^{(1)} = 0$. The induction is put in such an order that the right hand side of (36) is known from the previous work. It is directly integrated in the irrelevant case $4-2n<0$.

Lemma 4.3. For $n>2$, (36) has a unique solution with the properties (22), (23), (24), and (25). It is given by the convergent integral

$$\tilde{V}^{(r)}_{2n}(p_1,..., p_{2n-1}) = \int_0^1 \frac{dL}{L} L^{-4+2n} \tilde{K}^{(r)}_{2n}(Lp_1,..., Lp_{2n-1}) \qquad (37)$$

Proof. Equation (36) is equivalent to

$$L \frac{d}{dL} \{L^{-4+2n} \tilde{V}^{(r)}_{2n}(Lp_1,..., Lp_{2n-1})\} = L^{-4+2n} \tilde{K}^{(r)}_{2n}(Lp_1,..., Lp_{2n-1}) \qquad (38)$$

which is integrated to (37). The difference of two solutions to (38) satisfies the homogeneity condition

$$L \frac{d}{dL} \{L^{-4+2n} \Delta \tilde{V}^{(r)}_{2n}(Lp_1,..., Lp_{2n-1})\} = 0. \qquad (39)$$

Regularity at $p_1 = \cdots = p_{2n-1} = 0$ excludes solutions thereof other than zero. Therefore, (37) is unique. The other properties are obvious. ∎

The irrelevant part of the potential has thereby been determined. (37) is a recursion relation for the irrelevant vertices. Notice that (37) evaluates to zero in the case $n > r+1$. Therefore, the property (26) iterates to the next order.

Lemma 4.4. For $n > 2$, the vertices given by the integral (37), are independent of $\zeta^{(r-1)}$.

Proof. The differential vertex (34) is a linear function of $\zeta^{(r-1)}$ with

$$\begin{aligned}\frac{\partial}{\partial \zeta^{(r-1)}} \tilde{K}_{2n}^{(r)}(p_1,\dots, p_{2n-1}) &= -\binom{r}{2} \beta^{(2)} p_1^2 \delta_{n,1} - 2\binom{r}{1}\binom{2}{1} (\mu^{(1)} + \zeta^{(1)} p_1^2)\, p_1^2 \delta_{n,1} \\ &\quad - 2\binom{r}{1}\binom{4}{1} \tilde{C}(p_1 + p_2 + p_3)(p_1 + p_2 + p_3)^2 \delta_{n,2}\end{aligned} \tag{40}$$

zero for $n \geqslant 3$. The assertion follows by induction on n. ∎

We come to the non-irrelevant cases. We cannot integrate the differential equations (21) for the quadratic and the quartic vertex directly to (37). The non-negative scaling dimension, $4-2n \geqslant 0$, causes a divergence at $L=0$. This problem is cured by a Taylor expansion with remainder.

Consider first the quartic vertex. The differential equation (38) for the quartic vertex is

$$L \frac{\mathrm{d}}{\mathrm{d}L} \tilde{V}_4^{(r)}(Lp_1, Lp_2, Lp_3) = \tilde{K}_4^{(r)}(Lp_1, Lp_2, Lp_3) \tag{41}$$

Let us separate the coupling $\lambda^{(r)}$ from the quartic vertex according to

$$\tilde{V}_4^{(r)}(p_1, p_2, p_3) = \lambda^{(r)} + \int_0^1 \mathrm{d}L \frac{\mathrm{d}}{\mathrm{d}L} \tilde{V}_4^{(r)}(Lp_1, Lp_2, Lp_3) \tag{42}$$

Recall that we have fixed $\lambda^{(r)}$ by definition of g. Evaluate (41) at $L=0$, to conclude that the differential quartic kernel has to vanish at zero momentum. This condition determines $\beta^{(r)}$. The Taylor remainder can be computed from

$$\left\{L \frac{\mathrm{d}}{\mathrm{d}L} + 1\right\} \frac{\mathrm{d}}{\mathrm{d}L} \tilde{V}_4^{(r)}(Lp_1, Lp_2, Lp_3) = \frac{\mathrm{d}}{\mathrm{d}L} \tilde{K}_4^{(r)}(Lp_1, Lp_2, Lp_3) \tag{43}$$

obtained by taking one L-derivative of (42). The gain of one L-derivative is thus one unit of scaling dimension, whereupon we are back in the irrelevant case.

Lemma 4.5. The differential equation (41) has smooth solutions only if

$$\tilde{K}_4^{(r)}(0, 0, 0) = 0 \tag{44}$$

This condition is fulfilled if and only if

$$\beta^{(r)} = -\sum_{s=2}^{r-1} \binom{r}{s} \beta^{(s)} \lambda^{(r-s+1)} + \int \frac{d^4 q}{(2\pi)^4} \tilde{C}(q)\, \tilde{V}_6^{(r)}(0, 0, 0, q, -q) - \sum_{s=1}^{r-1} \binom{r}{s} 2 \binom{4}{1} \mu^{(s)} \lambda^{(r-s)} \tag{45}$$

where $\lambda^{(1)} = 1$.

I cannot resist from computing the second order coefficient $\beta^{(2)}$ at this instant. The six-point-vertex in second order is computed to[3]

$$\tilde{V}_6^{(2)}(p_1, \ldots, p_5) = -20 \left[\frac{1 - e^{-(p_1+p_2+p_3)^2}}{(p_1+p_2+p_3)^2} \right]_{\mathfrak{S}_5} \tag{46}$$

by means of (37), with $r = 2$ and $n = 3$. The integral in (45) of it is elementary. Using the value (32) of $\mu^{(1)}$ it follows that

$$\beta^{(2)} = \frac{-6}{(4\pi)^2} \tag{47}$$

which is the expected result. The negative sign has an important consequence. It tells that the flow on the renormalized ϕ^4-trajectory at weak coupling is asymptotically free in the infrared direction.

Lemma 4.6. (A) The differential equation (43) has a unique solution with the properties (22), (23), (24), and (25). It is given by the convergent integral

$$\frac{d}{dL} \tilde{V}_4^{(r)}(Lp_1, Lp_2, Lp_3) = \int_0^L \frac{dL'}{L'} L' \frac{d}{dL'} \tilde{K}_4^{(r)}(L'p_1, L'p_2, L'p_3) \tag{48}$$

[3] Consequently, $\tilde{V}_6^{(2)}(0, 0, 0, q, -q) = -8 - 12(1 - e^{-q^2}/q^2)$.

(B) The differential equation (41) has a unique solution with the properties (22), (23), (24), and (25). It is given by the convergent integral

$$\tilde{V}_4^{(r)}(p_1, p_2, p_3) = \lambda^{(r)} + \int_0^1 \frac{dL}{L} \tilde{K}_4^{(r)}(Lp_1, Lp_2, Lp_3) \tag{49}$$

Proof. The proof of (A) is the same as that of (37). One L-derivative is just enough to fall into the case of negative scaling dimension. Concerning (B), we notice that the integral (49) converges because

$$\tilde{K}_4^{(r)}(Lp_1, Lp_2, Lp_3) = O(L) \tag{50}$$

for all $(p_1, p_2, p_3) \in \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4$ due to the condition (44). The large momentum bound on (49) follows from the estimate

$$\| \tilde{V}_4^{(r)} \|_{\infty, \varepsilon} \leq |\lambda^{(r)}| + \frac{C_1}{\varepsilon} \sum_{i=1}^{3} \sum_{\mu=1}^{4} \left\| \frac{\partial}{\partial p_i^\mu} \tilde{V}_4^{(r)} \right\|_{\infty, \varepsilon} \tag{51}$$

where C_1 is a constant which is independent of r. The large momentum bound on the momentum derivatives of the quartic vertex follows from similar estimates. The other assertions are obvious. ∎

Notice that the large momentum bound is not uniform in ε. This is the price for the Taylor expansion. We then come to the quadratic vertex. Its personal differential equation reads

$$\left\{ L \frac{d}{dL} - 2 \right\} \tilde{V}_4^{(r)}(Lp) = \tilde{K}_2^{(r)}(Lp) \tag{52}$$

We represent it by a Taylor formula of order two with remainder. Because of Euclidean invariance, we have that

$$\tilde{V}_2^{(r)}(p) = \mu^{(r)} + \zeta^{(r)} p^2 + \frac{1}{2} \int_0^1 dL (1-L)^2 \frac{d^3}{dL^3} \tilde{V}_2^{(r)}(Lp) \tag{53}$$

We follow a similar procedure as in the case of the quartic vertex. The Taylor remainder is computed as solution to the differential equation

$$\left\{ L \frac{d}{dL} + 1 \right\} \frac{d^3}{dL^3} \tilde{V}_2^{(r)}(Lp) = \frac{d^3}{dL^3} \tilde{K}_2^{(r)}(Lp) \tag{54}$$

Three L-derivatives have brought us back to the irrelevant case.

Lemma 4.7. The differential equation (52) has smooth solutions (53) only if

$$-2\mu^{(r)}=\tilde{K}_2^{(r)}(0), \qquad 0=\frac{\partial}{\partial(p^2)}\tilde{K}_2^{(r)}(p)\bigg|_{p^2=0} \tag{55}$$

These conditions are fulfilled if and only if

$$\mu^{(r)}=\frac{1}{2}\left\{\sum_{s=2}^{r}\binom{r}{s}\beta^{(s)}\mu^{(r-s+1)} -\int\frac{d^4q}{(2\pi)^4}\tilde{C}(q)\,\tilde{V}_4^{(r)}(0,q,-q)+2\sum_{s=1}^{r-1}\binom{r}{s}\mu^{(s)}\mu^{(r-s)}\right\} \tag{56}$$

and

$$\zeta^{(r-1)}=\frac{-1}{\binom{r}{2}\beta^{(2)}+2\binom{r}{1}\binom{2}{1}\mu^{(1)}} \times\left\{\sum_{s=3}^{r}\binom{r}{s}\beta^{(s)}\zeta^{(r-s+1)}-\frac{\partial}{\partial(p^2)}\int\frac{d^4q}{(2\pi)^4}\tilde{C}(q)\,\tilde{V}_4^{(r)}(p,q,-q)\bigg|_{p^2=0} +\sum_{s=1}^{r-1}\binom{r}{s}\binom{2}{1}\mu^{(s)}\mu^{(r-s)}+\sum_{s=1}^{r-2}\binom{r}{s}\binom{2}{1}2\zeta^{(s)}\mu^{(r-s+1)}\right\} \tag{57}$$

The order r wave function $\zeta^{(r)}$ is a free parameter.

Notice that both $\mu^{(r)}$ and $\zeta^{(r-1)}$ are finite numbers. The integrals in (56) and (57) are convergent. Notice further that $\mu^{(r)}$ as given by (56), is independent of $\zeta^{(r-1)}$. The remaining work is easily put to order.

Lemma 4.8. (A) The differential equation (54) has a unique integral with the properties (23), (24), and (25). It is given by

$$\frac{d^3}{dL^3}\tilde{V}_2^{(r)}(Lp)=\int_0^L\frac{dL'}{L'}L'\frac{d^3}{dL'^3}\tilde{K}_2^{(r)}(L'p) \tag{58}$$

(B) The quadratic vertex assembled through (53), is unique up to the parameter $\zeta^{(r)}$. For any finite value of $\zeta^{(r)}$, it satisfies the properties (23), (24), and (25).

Proof. (A) is another application of the integral (37). (B) is put together from (A). The large momentum bound on the quadratic kernel follows from

$$\|\tilde{V}_2^{(r)}\|_{\infty,\varepsilon} \leqslant |\mu^{(r)}| + \frac{C_2}{\varepsilon}|\zeta^{(r)}| + \frac{C_3}{\varepsilon^2}\left\|\frac{\partial^2}{\partial(p^2)^2}\tilde{V}_2^{(r)}\right\|_{\infty,\varepsilon} \tag{59}$$

with some constants C_2 and C_3, both independent of r. Similar estimates hold for all momentum derivatives. ▮

The quadratic remainder depends on $\zeta^{(r-1)}$ but not on $\zeta^{(r)}$. The estimate for the quadratic kernel is valid for any finite value of $\zeta^{(r)}$.

Induction. We have shown that all assumptions of the induction hypothesis are valid to order r if they are valid up to order $r-1$. Since they are fulfilled to order one, they iterate to all orders of perturbation theory. The proof is complete.

5. β-FUNCTION

The β-function transforms under reparametrizations as a vector field. Consider reparametrizations of formal power series. It follows that $\beta^{(2)}$ and $\beta^{(3)}$ are universal, i.e., are not changed under reparametrizations. The other coefficients are not universal. We have showed that a β-function with finite coefficients exists for all choices of $\lambda(g)$. It is straight forward to determine the reparametrisation inductively order by order which brings all higher coefficients $\beta^{(r)}$, $r>3$, to zero. This is a canonical β-function for the renormalization group as a dynamical system. There is a direct implementation of this idea. Instead of imposing a condition on $\lambda(g)$ at the beginning we could have imposed a condition on $\beta(g)$, saying that it should be exactly cubic. Recall that $\lambda^{(1)}$ was normalized to one. It turns out that $\lambda^{(2)}$ can always be reparametrized to zero for the cubic β-function. Equation (45) is now used as follows. The second order equation determines $\beta^{(2)}$, the third order equation determines $\beta^{(3)}$, and the order $r+1$-equation, $r>3$, determines $\lambda^{(r)}$. Again all coefficients follow from convergent integrals and are hence finite.

6. CONCLUSIONS

The renormalization of Euclidean quantum fields to all orders of perturbation theory has been streamlined considerably by means of the renormalization group. We mention the work of Callan [C76] (using the field theoretic renormalization group), Gallavotti [G85], and Polchinski [P84] (using Wilson's renormalization group [WK74]). We also mention the subsequent contributions of Lesniewski [L83], Gallavotti and Nicolo [GN85], Hurd [H89], Keller, Kopper, and Salmhofer [KKS91].

The recursion relation furnished by (37), (45), (48), (55), (56), and (57), is the most direct renormalization scheme known to me. In the language of dynamical systems, we are computing an invariant curve in the center manifold of the trivial fixed point, whose tangent at the trivial fixed point is a (normal ordered) ϕ^4-vertex.

The renormalization group transformation (4) is a Gaussian convolution in rescaled form. An intrinsic scale is missing. When applied to a description of elementary particles, this scheme requires an additional datum: a renormalization scale. We have used a dimensionless formalism where all quantities are expressed in units of this renormalization scale.

It remains to be seen whether renormalization invariance is a solid starting point for non-perturbative studies.

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